$$d=\tfrac{1}{2}\|M^{-1/2}(Y-K)M^{-1/2}\|=\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}$$

$$\left[\sum_{q=1}^{n} m_{iq}^{-1/2}\sum_{p=1}^{n}(y_{qp}-k_{qp})m_{pj}^{-1/2}\right]^2 \tag{2}$$

Y is the desired corrected stiffness matrix, and K is a known approximate stiffness matrix obtained either by a finite-element method or by measurements. Here K is symmetric and can be singular if it includes rigid-body motions. Y must satisfy the constraints

$$Y\Phi=M\Phi\Omega^2 \tag{3}$$

$$Y=Y^T \tag{4}$$

where Ω^2 $(m\times m)$ is a diagonal matrix which represents the measured frequencies. The constraints must be necessary and sufficient for Φ and Ω^2 to be solutions of the dynamic equation

$$M\ddot{Z}+YZ=0 \tag{5}$$

Using Lagrange multipliers to include the constraints of Eqs. (3) and (4), the Lagrange function (ψ) is defined as follows:

$$\psi\equiv d+2\Pi\Lambda_y(Y\Phi-M\Phi\Omega^2)\Pi+\Pi\beta_y(Y-Y^T)\Pi \tag{6}$$

where

$$\Pi\Lambda_y(Y\Phi-M\Phi\Omega^2)\Pi=\sum_{i=1}^{n}\sum_{j=1}^{m}(\lambda_y)_{ij}\left(\sum_{r=1}^{n}y_{ir}\Phi_{rj}\right.$$

$$\left.-\sum_{r=1}^{n}m_{ir}\sum_{q=1}^{m}\Phi_{rq}\omega_{qj}^2\right) \tag{7a}$$

and

$$\Pi\beta_y(Y-Y^T)\Pi=\sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_y)_{ij}(y_{ij}-y_{ji}) \tag{7b}$$

Here Λ_y is a rectangular matrix of order $(n\times m)$, and β_y is an antisymmetric matrix of order $(n\times n)$.

$$\beta_y=-\beta_y^T \tag{8}$$

and

$$\omega_{pq}^2=\omega_{pp}^2 \quad \text{for } p=q \tag{9a}$$

$$\omega_{pq}^2=0 \quad \text{for } p\neq q \tag{9b}$$

where ω_{pp} are the measured frequencies.

The partial differentiation of ψ with respect to y_{ij} is set to zero, resulting in the y_{ij} for minimal ψ. In matrix form, one can get

$$\left[\frac{\partial\psi}{\partial y_{ij}}\right]=M^{-1}(Y-K)M^{-1}+2\Lambda_y\Phi^T+2\beta_y=0 \tag{10}$$

Adding Eq. (10) and its transpose to eliminate β_y yields

$$Y=K-M\Lambda_y\Phi^TM-M\Phi\Lambda_y^TM \tag{11}$$

Substituting Eq. (11) into Eq. (3) yields

$$M\Phi\Omega^2=K\Phi-M\Lambda_y-M\Phi\Lambda_y^TM\Phi \tag{12}$$

Rearranging Eq. (12) yields

$$M\Lambda_y=K\Phi-M\Phi\Omega^2-M\Phi\Lambda_y^TM\Phi \tag{13}$$

Taking the transpose of Eq. (13) yields

$$\Lambda_y^TM=\Phi^TK-\Omega^2\Phi^TM-\Phi^TM\Lambda_y\Phi^TM \tag{14}$$

Substitution of Eqs. (13) and (14) into Eq. (11) becomes

$$Y=K-K\Phi\Phi^TM+2M\Phi\Omega^2\Phi^TM+M\Phi\Lambda_y^TM\Phi\Phi^TM$$

$$-M\Phi\Phi^TK+M\Phi\Phi^TM\Lambda_y\Phi^TM \tag{15}$$

Again, Eqs. (13) and (14) are substituted into Eq. (15) to get

$$Y=K-K\Phi\Phi^TM-M\Phi\Phi^TK+2M\Phi\Phi^TK\Phi\Phi^TM$$

$$-M\Phi\Phi^TM\Lambda_y\Phi^TM-M\Phi\Lambda_y{}^TM\Phi\Phi^TM \tag{16}$$

Adding Eqs. (15) and (16) results in the solution of Y.

$$Y=K-K\Phi\Phi^TM-M\Phi\Phi^TK+M\Phi\Omega^2\Phi^TM+M\Phi\Phi^TK\Phi\Phi^TM \tag{17}$$

Conclusions

The stiffness matrix can be optimally corrected by using the Lagrange multiplier method. The corrected stiffness matrix Y can be directly obtained without making any assumptions, and the proof given is unique.

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Influence of Nonlinear Adhesive Behavior on Analysis of Cracked Adhesively Bonded Structures

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Introduction

ADHESIVE bonding is finding increased use in aerospace structures due to cost reduction and improved structural efficiency. The use of adhesive bonding in primary structures

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with greater reliability and confidence would require the development of analytical techniques and initial flaw requirements for bonded structures. This is required in order to meet damage-tolerance design criteria and to establish an allowable design stress for limiting the amount of slow crack growth.

The linear elastic fracture mechanics approach has been successfully used to predict crack growth in cracked bonded structures.[1-4] The analysis of cracked adhesively bonded laminated structures is discussed in Refs. 1 and 4-6. The majority of these analyses are based on the linear elastic behavior of the adhesive layer. It was indicated in Ref. 1 that the stresses in an adhesive layer will exhibit significant nonlinear behavior at the design limit stresses. This nonlinear adhesive behavior reduces the load transfer from the cracked layer to the uncracked layer due to large deformations of adhesive around the crack, resulting in large crack openings. This will significantly influence the stress intensity factors in the metallic layer and hence, the crack growth life of the structure. Thus, the influence of nonlinear behavior in adhesives such as FM-73 should be considered in establishing limit design stresses.

The analysis of cracked, adhesively bonded structures, assuming nonlinear adhesive behavior, has been performed by the finite-element methods.[4] Such an analysis requires long computer run times, and the cost would be prohibitive for parametric studies. Considering these factors, the mathematical method of analysis discussed in Refs. 1, 5, and 6 was modified to account for the nonlinear adhesive behavior.

Mathematical Formulation of the Problem

Consider the cracked adhesively bonded structure shown in Fig. 1. An initial through-the-thickness crack is assumed in material 1. There may be an initial debond present in the adhesive around the crack. It was shown in Refs. 1, 5, and 6 that the solution of the problem of Fig. 1 is reduced to the set of integral equations given by

$$\frac{h_a}{\mu_a}\tau_x(x,y) + \int_D\int [k_{11}(x,y;x_0,y_0)\tau_x(x_0,y_0) + k_{12}(x,y;x_0,y_0)\tau_y(x_0,y_0)]\mathrm{d}x_0\mathrm{d}y_0 = p_0 f_1(x,y) \tag{1a}$$

$$\frac{h_a}{\mu_a}\tau_y(x,y) + \int_D\int [k_{21}(x,y;x_0,y_0)\tau_x(x_0,y_0) + k_{22}(x,y;x_0,y_0)\tau_y(x_0,y_0)]\mathrm{d}x_0\mathrm{d}y_0 = p_0 f_2(x,y) \tag{1b}$$

where τ_x and τ_y are unknown shear stresses, k_{ij} are known kernels with logarithmic singularities,[5] f_1 and f_2 are known functions, p_0 is applied stress on crack surface in perturbation problem, and D is domain of integration.

The integral equations given by Eqs. (1) are solved numerically using collocation and reduced to a $2N\times 2N$ system of equations.[5,6]

An examination of Eqs. (1) shows that the kernels k_{ij} are not dependent upon the adhesive behavior. The adhesive shear modulus μ_a appears only outside of the integral sign in Eqs. (1). The solution of the shear stresses in the integral equations depends upon shear modulus μ_a. Equations (1) are solved first by assuming linear adhesive behavior and using the initial shear modulus. The shear stresses τ_x and τ_y are computed. If the shear stresses are within linear range of the adhesive stress-strain curve everywhere in D, the solution is correct. If not, new values of shear modulus μ_a are computed from the stress-strain curve at locations where the adhesive has exhibited nonlinear behavior. The analysis is carried out using these new values of shear moduli to compute new shear stresses. If these new stresses differ significantly from previous values, the iteration process is continued until the desired accuracy is obtained. It can be shown numerically that the shear stresses in domain D converge to within 1% accuracy in less than ten iterations.

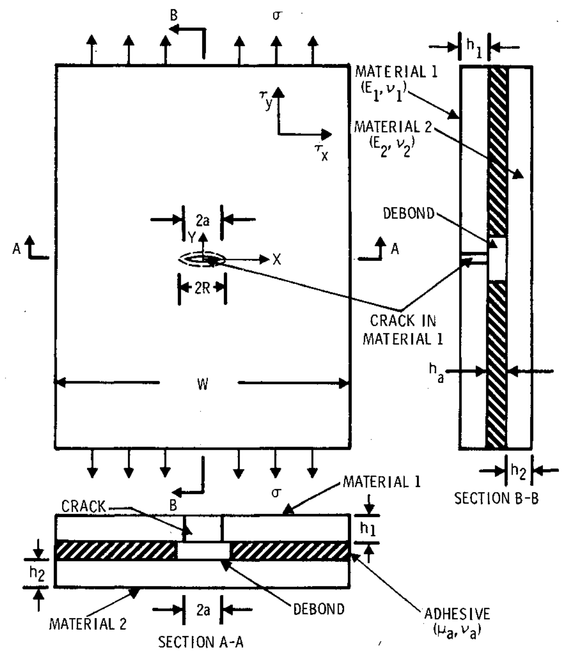

Fig. 1 Adhesively bonded structure with a debond and a crack.

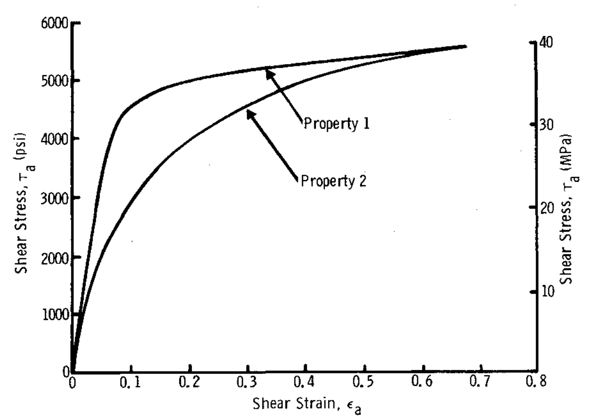

Fig. 2 Typical adhesive stress-strain relations used in a parametric study.

The iteration process discussed previously will require solving a set of $2N\times 2N$ simultaneous equations several times, resulting in long computer run times. During the course of this investigation, it was found that the value of shear stress τ_x is small and its effect can be neglected. Neglecting the influence changed the shear stress τ_y and stress intensity factors in the cracked layer by less than 2%. Hence, in the analysis the influence of τ_x was neglected. Equation (1a) is put equal to zero. This condition leads to the assumption that displacement of two metallic layers in x direction is the same. Equations (1) reduce to

$$h_a\epsilon_y(x,y) + \int_D\int k_{22}(x,y;x_0,y_0)\tau_y(x_0,y_0)\mathrm{d}x_0\mathrm{d}y_0 = p_0 f_2(x,y) \tag{2}$$

where $\epsilon_y(x,y)$ is the strain corresponding to shear stress $\tau_y(x,y)$. This will require only $N\times N$ simultaneous equations to be solved. A computer program is written to solve Eq. (2) numerically, and the iteration processes are carried out until the difference in the normalized stress intensity factor, $(k/p_0\sqrt{\pi a})$ obtained by two consecutive iterations is less than

Table 1 Influence of adhesive behavior on stress intensity factors

	$K/p_0\sqrt{\pi a}$, adhesive property 1			$K/p_0\sqrt{\pi a}$, adhesive property 2		
Half-crack length, mm	Linear	Nonlinear	Percent difference	Linear	Nonlinear	Percent difference
2.54	0.8589	0.8590	0.01	0.8418	0.8796	4.49
7.62	0.5958	0.6050	1.56	0.5739	0.6583	14.71
12.70	0.4747	0.4867	2.51	0.4559	0.5389	18.21
17.78	0.4055	0.4173	2.90	0.3890	0.4662	19.84
22.86	0.3595	0.3707	3.10	0.3446	0.4163	20.81

Table 2 Influence of adhesive thickness on stress intensity factors

Half-crack length, a (mm)	Adhesive thickness ha (mm)	$K/p_0\sqrt{\pi a}$ Linear	$K/p_0\sqrt{\pi a}$ Nonlinear	Percent difference
2.54	0.102	0.7644	0.8362	9.39
	0.203	0.8418	0.8796	4.49
	0.305	0.8797	0.9024	2.59
7.62	0.102	0.4918	0.6007	22.16
	0.203	0.5739	0.6583	14.71
	0.305	0.6251	0.6938	11.00
12.70	0.102	0.3871	0.4866	25.69
	0.203	0.4559	0.5389	18.21
	0.305	0.5005	0.5731	14.52
17.78	0.102	0.3289	0.4191	27.42
	0.203	0.3890	0.4662	19.84
	0.305	0.4283	0.4974	16.15
22.86	0.102	0.2904	0.3734	28.56
	0.203	0.3446	0.4163	20.81
	0.305	0.3800	0.4449	17.07

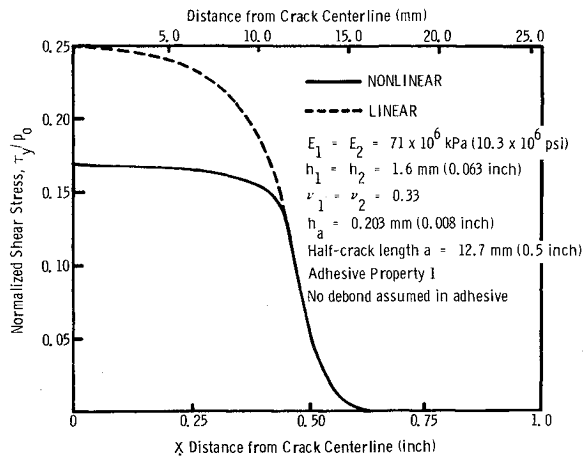

Fig. 3 Distribution of shear stresses along crack plane ($y = 0$); applied stress $p_0 = 191$ MPa (27.7 ksi).

0.0005. It was found that the solution generally converges in four to five iterations.

Example

The two-layer, adhesively bonded structure shown in Fig. 1 (each layer of 7075-T6 aluminum was 1.6 mm thick) was analyzed, assuming the two different nonlinear adhesive properties shown in Fig. 2. The stress intensity factors obtained for linear and nonlinear adhesive behaviors are shown in Table 1 for the two adhesive properties (adhesive thickness = 0.203 mm, applied stress = 191 MPa). It is seen that the influence of nonlinear behavior on stress intensity factors is small for adhesive property 1; however, for adhesive property 2, the influence of nonlinear behavior is significant.

The influence of adhesive thickness on stress intensity factors for linear and nonlinear adhesive behavior (adhesive property 2) is shown in Table 2 for various crack lengths at an applied stress of 191 MPa. The table also shows the percentage difference in stress intensity factors for linear and nonlinear adhesive behavior. It is seen that the influence of nonlinearity increases with the increase in crack length for a fixed adhesive thickness due to an increase in load transfer through the adhesive layer. The table also indicates that the influence of nonlinearity is large for smaller adhesive thicknesses. This is because a reduced adhesive thickness causes more load transfer to the uncracked layer by reducing the opening of the crack in the cracked layer.

The variation of normalized shear stress at the crack plane as a function of distance from the centerline of the crack is shown in Fig. 3 for linear and nonlinear adhesive behavior. The normalized shear stresses for nonlinear behavior are considerably lower than those for linear behavior.

Conclusions

1) The nonlinear adhesive behavior causes less load transfer to the uncracked layer, resulting in increased stress intensity factors in the cracked layer. The larger the nonlinearity shown by the adhesive, the larger the increase in the stress intensity factors.

2) The influence of nonlinear adhesive behavior on stress intensity factors decreases with an increase in bondline

thickness. An increase in bondline thickness causes a reduction in load transfer to the uncracked layer, and hence, reduced shear stresses. This results in a reduced effect of adhesive nonlinearity.

3) The maximum normalized shear stresses in the adhesive are considerably reduced if the nonlinear adhesive behavior is taken into consideration.

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